

Quasi-Splines and their Moduli

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March 27, 2014

Abstract

We study what we call **quasi-spline sheaves** over locally Noetherian schemes. This is done with the intention of considering splines from the point of view of moduli theory. In other words, we study the way in which certain objects that arise in the theory of splines can be made to depend on parameters. In addition to **quasi-spline sheaves**, we treat **ideal difference-conditions**, and individual **quasi-splines**. Under certain hypotheses each of these types of objects admits a fine moduli scheme. The moduli of quasi-spline sheaves is proper, and there is a natural compactification of the moduli of ideal difference-conditions. We include some speculation on the uses of these moduli in the theory of splines and topology, and an appendix with a treatment of the **Billera-Rose homogenization** in scheme theoretic language.

1 Introduction

1.1 Quasi-spline sheaves

Given a scheme B , the s -fold sum \mathcal{O}_B^s is a sheaf of \mathcal{O}_B -algebras with entrywise multiplication and addition. A quasi-coherent subsheaf \mathcal{S} of \mathcal{O}_B^s generalizes the notion of spline functions on B if it contains the diagonal copy of $\mathcal{O}_B \subseteq \mathcal{O}_B^s$ and is closed under multiplication. This is simply to say \mathcal{S} is a quasi-coherent \mathcal{O}_B -subalgebra of \mathcal{O}_B^s . Because of the close relationship with spline functions, we call such an \mathcal{S} a sheaf of **quasi-splines**. A simple example is Example 1.1.

Example 1.1. Let $B = \operatorname{Spec} \mathbf{R}[x]$. The sheaf associated to the $\mathbf{R}[x]$ -module

$$S = \{(g_1, g_2) \mid g_1 - g_2 \in (x^2)\} \subseteq (\mathbf{R}[x])^2$$

is a sheaf of quasi-splines. It is naturally thought of as the splines with continuous first derivatives over the subdivision $\mathbf{R} = (-\infty, 0] \cup [0, \infty)$.

We will focus on quasi-splines over projective schemes as in Example 1.2.

Example 1.2. Let $B = \mathbf{P}_R^1 = \text{Proj } R[x, z]$. The sheaf \mathcal{S} associated to the graded $R[x, z]$ -module

$${}^hS = \{(G_1, G_2) \mid G_1 - G_2 \in (x^2)\} \subseteq (R[x, z])^2.$$

is a sheaf of quasi-splines. hS is the homogenization of the splines of Example 1.1 as defined in [BR91]. Additionally, hS is saturated, i.e. the map

$${}^hS \rightarrow \bigoplus_d \Gamma(\mathbf{P}_R^1, \mathcal{S}(d))$$

is a graded isomorphism. Together, these facts imply the module from Example 1.1 is canonically identified as $S = \mathcal{S}(U_0)$ where $U_0 \subseteq \mathbf{P}_R^1$ is the set on which $z \neq 0$.

Although quasi-splines are closely related to splines. It is not always possible to think of them as such. Consider Example 1.3.

Example 1.3. Let $B = \text{Spec } R[x]$. The sheaf associated to the $R[x]$ -module

$$S = \{(g_1, g_2) \mid g_1 - g_2 \in (x^2 + 1)\} \subseteq (R[x])^2$$

is a sheaf of quasi-splines, but it cannot be thought of as splines in any obvious way.

We are interested in studying quasi-spline sheaves which depend of parameters. To this end, given a Z -scheme B we define a **Z -family of quasi-spline sheaves over B** as

- a sheaf of quasi-splines \mathcal{S} over a B such that
- for any morphism $f: Z' \rightarrow Z$, the pullback $\pi_B^* \mathcal{S}$ is a sheaf of quasi-splines over $Z' \times_Z B$.

This definition eliminates from consideration sheaves $\mathcal{S} \subseteq \mathcal{O}_B^s$ whose inclusion map $\mathcal{S} \rightarrow \mathcal{O}_B^s$ fails to be an inclusion after fixing the value of the parameters. Example 1.4 gives a sheaf of quasi-splines which fails to be a family.

Example 1.4. Let $Z = \text{Spec } R[z]$, and take $B = \text{Spec } R[x]$ as in Example 1.1. The morphism $B \rightarrow Z$ is given by $z \mapsto x$. The sheaf associated to the $R[x]$ -module

$$S = \{(g_1, g_2) \mid g_1 - g_2 \in (x^2)\} \subseteq (R[x])^2$$

is not a Z -family. This can be seen by first setting $g = (-x^2, x^2)$ and identifying

$$S = R[x][g]/(g^2 - x^4).$$

The map $S \rightarrow (R[x])^2$ sends $a_0 + a_1 g \mapsto (a_0 - a_1 x^2, a_0 + a_1 x^2)$. So when $z = 0$ we have $x = 0$ and $S|_{z=0} = R[g]/(g^2)$. The map to $(R[x])^2|_{z=0} \cong R^2$ is not an inclusion since it sends $g \mapsto 0$.

On the other hand, Example 1.5 shows some quasi-spline sheaves are indeed families.

Example 1.5. Let $Z = \text{Spec } \mathbf{R}[z]$, and $B = \text{Spec } \mathbf{R}[z][x, y]$. The sheaf associated to the $\mathbf{R}[z][x, y]$ -module

$$S = \{(g_1, g_2, g_3) \mid \begin{array}{l} g_1 - g_2 \in (x), \\ g_2 - g_3 \in (y), \\ g_1 - g_3 \in (x + y - z) \end{array}\} \subseteq (\mathbf{R}[z][x, y])^3.$$

is a Z -family. One can check this by observing that as an $\mathbf{R}[z]$ -module, S has a free basis whose \mathbf{R} -span is

$$\mathbf{R}[x, y] \cdot \nu_0 \oplus \mathbf{R}[x, y] \cdot \nu_1 \oplus \mathbf{R}[x, y] \cdot \nu_2 \oplus \mathbf{R}[y] \cdot \nu_3$$

where

$$\begin{aligned} \nu_0 &= (1, 1, 1), \\ \nu_1 &= (0, zx - x^2, zx - x^2 - xy), \\ \nu_2 &= (0, 0, zy - xy - y^2), \text{ and} \\ \nu_3 &= (0, xy, 0). \end{aligned}$$

The definition of families of quasi-spline sheaves guarantees that if we fix a scheme Y over T , the assignment

$$\mathcal{QS}^{(s)}(Y/T)(Z) = \{Z\text{-families of quasi-spline sheaves } \mathcal{S} \subseteq \mathcal{O}_{Z \times_T Y}^s\}$$

is functorial for T -schemes Z . This means there is some hope that one can find a representing scheme, i.e. there is a **moduli scheme** $QS^{(s)}(Y/T) \in T\text{-schemes}$ such that $\text{Mor}(Z, QS^{(s)}(Y/T)) = \mathcal{QS}^{(s)}(Y/T)(Z)$. Our first theorem is on the existence of this moduli scheme.

Theorem 3.10. *In the category of locally Noetherian schemes, for a flat, projective T -scheme Y the functor*

$$\mathcal{QS}^{(s)}(Y/T): (T\text{-schemes})^{op} \rightarrow \mathbf{Sets}$$

is representable by a closed subscheme $QS^{(s)}(Y/T)$ of the Quot scheme $\text{Quot}(\mathcal{O}_Y^s/Y/T)$.

1.2 Ideal difference-conditions

In many of the applications we have in mind, \mathcal{S} is defined as the subset of \mathcal{O}_Y^s whose sections satisfy **ideal difference-conditions**. That is to say, the sheaf \mathcal{S} is **defined by conditions** that written locally are

$$\mathcal{S} = \{(g_1, \dots, g_s) \mid g_j - g_k \in \mathfrak{I}_{jk} \text{ for } 1 \leq j < k \leq s\} \quad (1)$$

for $\binom{s}{2}$ ideals $\mathfrak{I}_{jk} \subseteq \mathcal{O}_Y$. All of our examples defined quasi-splines this way.

Allowing the ideals to vary by introducing parameters leads to an interesting subtlety. For a fixed value of the parameters, there are two different ways to define a quasi-spline sheaf. On one hand we can compute the sheaf of quasi-splines defined by the ideals with the parameters considered as variables, and then restrict the sheaf to the fixed parameter values. On the other, we could fix value of the parameters in the ideals and then compute a possibly different quasi-spline sheaf.

To be clear, denote by $\mathcal{S}_{\mathfrak{J}}$ the sheaf of quasi-splines defined by ideals $(\mathfrak{J}_{jk})_{jk}$. For simplicity assume that we have a single parameter $z \in \mathbf{R}[z]$, and we are interested in the fixed value $z = 0$. Consider ideals $(\mathfrak{J}_{jk}(z))_{jk}$ which depend on z . There is a natural map

$$\mathcal{S}_{\mathfrak{J}(z)}|_{z=0} \rightarrow \mathcal{S}_{\mathfrak{J}(z=0)} \quad (2)$$

which may or may not be an isomorphism. However, the map is an inclusion for all z if and only if $\mathcal{S}_{\mathfrak{J}(z)}$ is a Z -family of quasi-spline sheaves (here $Z = \text{Spec} \mathbf{R}[z]$).

The ideals in Example 1.5 are shown in the continuation of this example to lead to sheaves where the map in (2) is an inclusion but not an isomorphism at $z = 0$.

Example 1.5 continued. For any given $z \in \mathbf{R}$, the quasi-splines of Example 1.5 are naturally thought of as splines over the region Ω of plane in the complement of the triangle with vertices $(z, 0)$, $(0, z)$, and $(0, 0)$. The relevant subdivision is shown in Figure 1, and is made up of three parts

- $\Omega_1 = \{(x, y) \mid 0 \leq x, 0 \leq x + y - z\}$,
- $\Omega_2 = \{(x, y) \mid x \leq 0 \leq y\}$, and
- $\Omega_3 = \{(x, y) \mid x + y - z \leq 0, y \leq 0\}$.

The sheaf defined by first setting $z = 0$ and then computing quasi-splines is strictly larger than those obtained by restricting from the family. For instance,

$$(y, y - x, -x)$$

is a quasi-spline for the $z = 0$ ideals, but it is not the restriction of a quasi-spline in the family. In other words, the map in (2) is an inclusion but is not surjective.

As in Example 1.1 these two sets of splines can be characterized in terms of continuity and the existence of derivatives. The splines in the family when restricted to $z = 0$ are exactly those which are both continuous over Ω and have continuous first partial derivatives at $(0, 0) \in \mathbf{R}^2$. The splines computed by first setting $z = 0$ is the larger set of all continuous splines.

With these considerations in mind, for a Z -scheme B we define a **Z -family of ideal difference-conditions over B** to be an $\binom{s}{2}$ -tuple $(\mathfrak{J}_{jk})_{jk}$ of quasi-coherent ideals $\mathfrak{J}_{jk} \subseteq \mathcal{O}_B$ which have two properties:

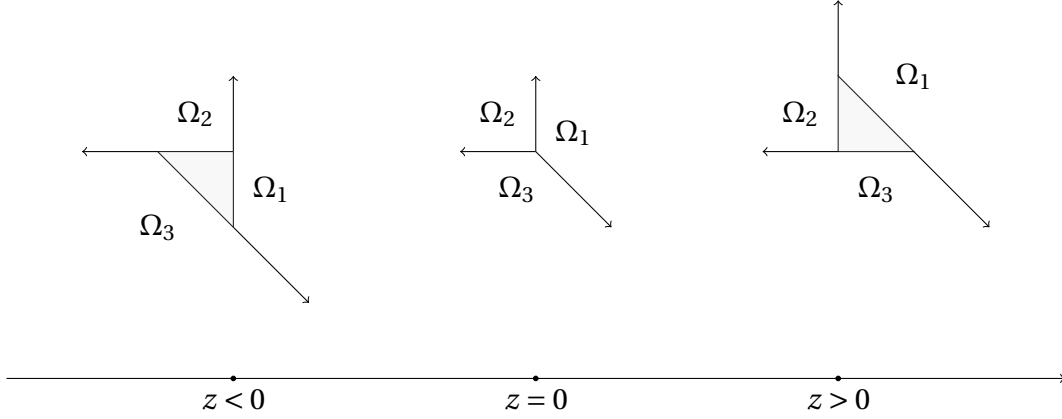


Figure 1: Ω and the subdivision as z varies from Example 1.5 and its continuation.

- \mathfrak{I}_{jk} remains an ideal after any base change, i.e. $\mathbf{V}(\mathfrak{I}_{jk})$ is flat, and
- base change of the quasi-splines defined by the \mathfrak{I}_{jk} 's equals the quasi-splines defined by the base change of the \mathfrak{I}_{jk} 's.

Denoting the Hilbert scheme by $\mathrm{Hilb}(Y/T)$, we have the representability theorem:

Theorem 4.4. *In the category of locally Noetherian schemes, for any projective T -scheme Y the functor*

$$\mathcal{C}^{(s)}(Y/Z)(Z) = \{Z\text{-families of ideal difference-conditions on } Z \times_T Y\}$$

is representable by a scheme $C^{(s)}(Y/Z)$ obtained as the universal flattening stratification of a certain sheaf \mathcal{H} over $\mathrm{Hilb}(Y/T)^{(s)}$.

Notice that there is no T -flatness condition on Y in this statement. This has the interesting consequence that $C^{(s)}(Y/Z)$ exists, even if $QS^{(s)}(Y/Z)$ might not.

The moduli of ideal difference conditions is not proper. This can be seen in Example 1.5. This example suggests a **compactification** of $C^{(s)}(Y/T)$. To do this we use an auxiliary scheme based on the notion of a **compatible pair** $(\mathcal{S}, (\mathfrak{I}_{jk})_{jk})$. By this we mean the composition

$$\mathcal{S} \rightarrow \mathcal{O}_B^s \rightarrow \bigoplus_{jk} \mathcal{O}/\mathfrak{I}_{jk}$$

is zero. Equivalently, \mathcal{S} is contained in the quasi-spline sheaf $\mathcal{S}_{\mathfrak{I}}$ made up of sections of \mathcal{O}_B^s which satisfy the ideal difference-conditions $(\mathfrak{I}_{jk})_{jk}$. However, it is not necessary that \mathcal{S} equals $\mathcal{S}_{\mathfrak{I}}$.

Based on this idea, we construct the **moduli of compatible pairs** $P^{(s)}(Y/T)$. This scheme is proper and in the category of locally Noetherian schemes it represents the functor

$$\mathcal{P}^{(s)}(Y/T)(Z) = \{\text{compatible pairs } (\mathcal{S}, (\mathfrak{I}_{jk})_{jk}) \text{ of } Z\text{-families over } Z \times_T Y\}.$$

$C^{(s)}(Y/Z)$ sits as a locally closed subscheme of $P^{(s)}(Y/T)$ and presented this way, a natural compactification is given by the scheme-theoretic closure $\overline{C}^{(s)}(Y/T)$ of $C^{(s)}(Y/Z)$ in $P^{(s)}(Y/T)$.

An important property of the compactification is that it allows for a **universal family of quasi-spline sheaves** over $\overline{C}^{(s)}(Y/T) \times_T Y$ that extends the one naturally living over $C^{(s)}(Y/Z) \times_T Y$. The case for the “correctness” of this choice of compactification can be made on the grounds of Proposition 4.10:

Proposition 4.10. *Let $C^{(s)}(Y/Z) \rightarrow H$ be a morphism to a scheme such that $H \times_Y Y$ is equipped with an H -family of compatible pairs whose restriction to $C^{(s)}(Y/Z) \times_T Y$ is the universal family of ideal difference-conditions. Assume H equals the scheme theoretic image of $C^{(s)}(Y/Z)$ in H . Then the morphism $H \rightarrow P^{(s)}(Y/T)$ factors through $\overline{C}^{(s)}(Y/T)$.*

However, a possibly more compelling fact is the naturality of the families $\overline{C}^{(s)}(Y/T)$ admits, such as the one in Example 1.5.

On the moduli of ideal-difference conditions the Hilbert polynomial of the quasi-spline sheaf is locally unchanged. It is natural to ask for a further stratification of the moduli space into subschemes on which the full Hilbert series is unchanged. Our section on ideal difference-conditions concludes with a discussion on how the **degeneracy loci** of a morphism of certain locally free sheaves can be used to give such a stratification.

1.3 Quasi-splines

In the construction of the moduli space $QS^{(s)}(Y/T)$ we assumed that Y was flat. A consequence of this is that the Hilbert polynomial of \mathcal{S} is locally independent of the point in $QS^{(s)}(Y/T)$. Using this fact, we get another interesting theorem about representing the functor of sections:

Theorem 5.8. *Fix flat T -scheme Y , and for notational simplicity, assume that Hilbert polynomial $p_{\mathcal{O}_Y}$ of Y is independent of T . In the category of locally Noetherian schemes, the functor*

$$\mathcal{E}_{p,d}^{(s)}(Y/T)(Z) = \{\tau \in \Gamma(Z \times_T Y, \mathcal{S}(d)) \mid \mathcal{S} \text{ has Hilbert polynomial } p\}$$

is representable for $d \geq m$ where m depends on p and $p_{\mathcal{O}_Y}$.

Denote the universal quotient associated to the Quot scheme by \mathcal{G} . Over the piece of $\text{QS}^{(s)}(Y/T)$ which lies in the component of the Quot scheme labeled by the polynomial $p_{\mathcal{G}} = s p_{\mathcal{O}_Y} - p$, the representing scheme is

$$E_{p,d}^{(s)}(Y/T) = \text{Spec Sym } \mathcal{V}_d,$$

where

$$\mathcal{V}_d = \mathcal{H}om_{\text{QS}^{(s)}(Y/T)}(\pi_* \mathcal{S}(d), \mathcal{O}_{\text{QS}^{(s)}(Y/T)}),$$

and $\pi: \text{QS}^{(s)}(Y/T) \times_T Y \rightarrow \text{QS}^{(s)}(Y/T)$ is the projection. The number m can be taken to be the maximum the Gotzmann numbers of the pair of polynomials given below in Lemma 5.7.

A fact of independent interest used in the proof is that for d at or beyond this value, the sheaf $\pi_* \mathcal{S}(d)$ is locally free. This implies that the rank of $\pi_* \mathcal{S}(d)$ agrees with its Hilbert polynomial.

1.4 Billera-Rose Homogenization

The paper concludes with an appendix on the homogenization procedure introduced in [BR91]. Originally, this was an identification between splines on a triangulation in \mathbf{R}^n with splines on the cone over the triangulation in \mathbf{R}^{n+1} . The splines over the cone form a graded module, and the degree d homogeneous piece of this module is naturally identified with splines on \mathbf{R}^n all of whose entries are degree $\leq d$.

We consider this procedure as a comparison between quasi-spline sheaves on three schemes: the original scheme A , its projective closure \hat{A} , and the affine cone over the projective closure. We find in Proposition A.11 that if the homogeneous coordinate ring \hat{A} is a quotient of the homogeneous coordinate ring of the ambient projective space, then Billera-Rose homogenization translates questions about quasi-splines on the original scheme into questions about an quasi-splines on its projective closure.

To prove this result, homogenization and projective closure is formulated in terms of filtered algebras and modules, rather than the traditional approach of submodules of graded modules [Gro61]. We find that this approach is very satisfying and interesting in its own right.

1.5 Remarks and Speculations

1.5.1 The questions of dimension and flatness

The constructions here are particularly suitable to the **dimension question** in the theory of splines. This was posed by Strang [Str74], and in this context asks

“What is the Hilbert series of \mathcal{S} ?”

The Hilbert polynomial of \mathcal{S} does not change as one moves around within connected components of $C^{(s)}(Y/T)$. This means that just knowing the connected component determines most of the Hilbert series.

In general, the problem of determining the initial terms of the Hilbert series is daunting. However, when cohomology commutes with base change for $\mathcal{O}_{C^{(s)}(Y/T) \times_T Y}$ and the $\mathcal{O}_{C^{(s)}(Y/T) \times_T Y} / \mathfrak{I}_{jk}$ (e.g. hypersurface ideal difference-conditions on \mathbf{P}^n) the geometry governing the rank of $\Gamma(Y, \mathcal{S}(d))$ for small d is the stratification of $C^{(s)}(Y/T)$ defined by the **degeneracy loci** of the map

$$\Gamma(Y, \mathcal{O}_{C^{(s)}(Y/T)}^s(d)) \rightarrow \bigoplus_{jk} \Gamma(Y, \mathcal{O}_{C^{(s)}(Y/T) \times_T Y}(d) / \mathfrak{I}_{jk}(d)).$$

This is proved below in Proposition 4.12. Ultimately, the dimension question for small d is a question of understanding how these subschemes lie in $C^{(s)}(Y/T)$.

A related question concerns the **flatness** of the splines over Y :

“Which quasi-spline sheaves are flat \mathcal{O}_Y -modules?”

This question was posed by Billera and Rose in the context of the dimension question [BR92]. This can be interpreted in terms of the s^{th} **Fitting subscheme** Z_s of the universal quasi-spline sheaf on $C^{(s)}(Y/T) \times_T Y$. Recall that Z_s is the closed subscheme over which \mathcal{S} cannot be generated by s -sections. The image of Z_s under the projection $\pi: C^{(s)}(Y/T) \times_T Y \rightarrow C^{(s)}(Y/T)$ is made up of those quasi-spline sheaves which are not flat on Y .

These moduli spaces, their geometry and arithmetic, complement a larger line of investigation into multivariate splines. Bernstein-Bézier methods as well as tools from commutative algebra, algebraic geometry, homological algebra, and symbolic computation, have produced many results; for example [MS75, CW83, Sti83, Bil88, BR91, Ros95, SS97, GS98, MS09, Toh05, MV13, Sch14]. These give a lot of information about the moduli scheme, the relevant degeneracy loci and Fitting subschemes. We expect that as we learn more about its geometric and arithmetic properties, new dimension formulas and flatness criteria will reveal themselves.

1.5.2 Spline domains and approximation strategies

The existence of these moduli spaces points to some interesting possibilities in approximation theory. For instance in an approximation or interpolation problem, rather than fixing a sheaf \mathcal{S} of quasi-splines and trying to find a best candidate in $\Gamma(Y, \mathcal{S}(d))$, one could consider the problem of finding a best quasi-spline in $E_{p,d}^{(s)}(Y/T)$. In principle, this frees one from committing to a fixed **spline domain** $D: \Omega = \Omega_1 \cup \dots \cup \Omega_s \subseteq Y(\mathbf{R})$, and allows the subdivision to vary.

Putting this onto a satisfactory mathematical footing would require a moduli of spline-domains \mathcal{D} . One could then consider compatible triples

$$(D, (\mathfrak{J}_{jk})_{jk}, \tau) \in \mathcal{D} \times_{\mathbf{T}} \overline{\mathbf{C}}^{(s)}(Y/T) \times_{\mathbf{T}} E_{p,d}^{(s)}(Y/T).$$

We know of no such object \mathcal{D} in the literature, but see no reason why it shouldn't exist. Some insight is provided by Example 1.6 which indicates the sort of phenomena that arise when interpreting quasi-splines as splines.

Example 1.6. Consider $f_z(x, y) = (z^2 - 1)y - z(x^2 + y^2 - 1)$ as a family of polynomials on \mathbf{R}^2 parameterized by $z \in [-1, 1]$. For each z write $\Omega = \Omega_1^z \cup \Omega_2^z \subseteq \mathbf{R}^2$ where

- $\Omega_1^z = \{(x, y) \mid f_z(x, y) \leq 0\}$, and
- $\Omega_2^z = \{(x, y) \mid f_z(x, y) \geq 0\}$.

Consider the family of splines defined by the quasi-spline

$$g_z = (-zf_z(x, y), zf_z(x, y)).$$

Observe that at both $z = -1$ and $z = 1$ the quasi spline is $(x^2 + y^2 - 1, 1 - x^2 - y^2)$. However, Ω_1 and Ω_2 have switched, so the spline has reversed signs. This is illustrated in Figure 2. Topologically, this is an interval with distinct endpoints in $\mathcal{D} \times E_{p,d}^{(s)}(Y/T)$ whose projection to $E_{p,d}^{(s)}(Y/T)$ is a loop.

1.5.3 Topology

In addition to the close relationship to spline theory, quasi-splines appear in equivariant cohomology and equivariant intersection homology [GPT13] under the name **generalized splines**. This opens up a huge area of connections to topics such as geometric representation theory, Schubert calculus, and quantum cohomology.

2 Assumptions, Conventions and Notations

This paper is written in scheme-theoretic language. In this section we collect several relevant standard results, make notations, and specify our assumptions. These break-down roughly as notations for projective geometric constructions, results relevant to cohomology and base change, and finally the representability of certain functors such as flattening stratifications and Quot schemes.

Assumption. We fix an integer $s \geq 1$ throughout. We are working a category of locally Noetherian schemes, and if we are over a base scheme, this scheme is also locally Noetherian. We fix schemes Y and T . This allows us to simplify our notation. For example, $QS^{(s)}(Y/T)$ will be written QS .

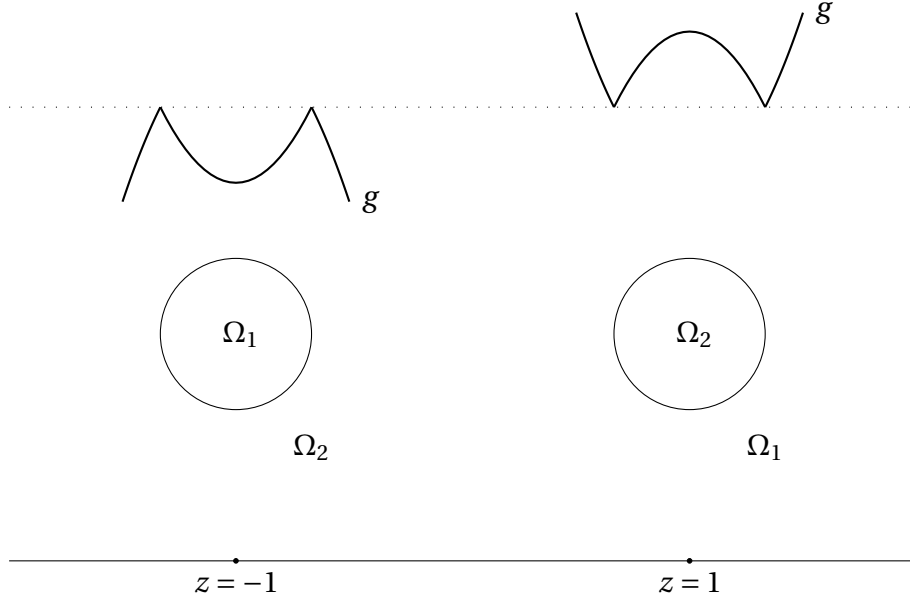


Figure 2: The splines and spline domains at the ends of the interval $[-1, 1]$ from Example 1.6. The spline is indicated by its graph when restricted to $y = 0$. These two functions on \mathbf{R}^2 are defined by the same quasi-spline.

Notation. If $B \rightarrow Z$ is a Z -scheme, \mathcal{F} is a sheaf on B , and $\phi: Z' \rightarrow Z$ a morphism, we denote

- the fiber product $B_{Z'} = B \times_Z Z'$, and
- the pullback $\phi^* \mathcal{F}$ on $B_{Z'}$ by $\mathcal{F}|_{Z'}$.

A point $q \in Z$ is assigned the scheme structure $\text{Spec} \mathbf{k}(q)$, and we often write B_q and $\mathcal{F}|_q$ with this scheme structure on q assumed. The vertical bar in the notation for the pullback is to avoid confusion with the stalk \mathcal{F}_b of \mathcal{F} at a point $b \in B$.

2.1 Projective Geometry

We review here some basic constructions and facts of projective geometry. This is done mostly to establish notation.

Serre's Generation and Finiteness Theorems. ([Ser55] see also [Har77, Theorem II.5.17]) Let Z be a Noetherian scheme and \mathcal{F} a coherent sheaf on a projective Z -scheme $\pi: B \rightarrow Z$. Then

- $R^i \pi_* \mathcal{F}$ is a coherent \mathcal{O}_Z -module, and

for all sufficiently large d

- $\mathcal{F}(d)$ is generated by global sections, and
- $R^i \pi_* \mathcal{F}(d) = 0$.

Notation. For a sheaf of graded modules N over a sheaf of graded \mathcal{O}_Z -algebras \mathcal{R} , we write \tilde{N} for the associated sheaf on $\text{Proj}(\mathcal{R})$. Conversely, given a sheaf \mathcal{F} on $\text{Proj}(\mathcal{R})$, we write $\Gamma_*(\mathcal{F})$ for the graded $\Gamma_*(\mathcal{O}_{\text{Proj}(\mathcal{R})})$ -module

$$\Gamma_*(\mathcal{F}) = \bigoplus_d \pi_* \mathcal{F}(d),$$

and $\Gamma_{\geq m}(\mathcal{F})$ if we only take those $d \geq m$. Here $\pi: \text{Proj}(\mathcal{R}) \rightarrow Z$ is the projection.

Lemma 2.1. *The following statements describe the relationship between Γ_* , $\Gamma_{\geq m}$ and $\tilde{(\cdot)}$:*

- $\tilde{(\cdot)}$ is exact;
- Γ_* and $\Gamma_{\geq m}$ are left exact;
- $\tilde{(\cdot)} \circ \Gamma_{\geq m} = \tilde{(\cdot)} \circ \Gamma_*$;
- $\tilde{(\cdot)}$ left adjoint to Γ_* ;
- the counit $\epsilon: \tilde{(\cdot)} \circ \Gamma_* \rightarrow \mathbf{1}$ is a natural isomorphism;
- the unit $\eta: \mathbf{1} \rightarrow \Gamma_* \circ \tilde{(\cdot)}$ is called the **saturation map**.

Proof. Omitted. □

Remark 2.2. In the affine case, $\tilde{(\cdot)}$ is used for the functor assigning to a module over a ring the associated sheaf on the spectrum. Its adjoint equivalence is $\Gamma(\cdot)$.

The Cohomology of Projective Space. ([Har77, Theorem III.5.1]) Let Z be an affine Noetherian scheme. Then:

- the natural map $\mathcal{O}_Z[x_0, \dots, x_n] \rightarrow \Gamma_*(\mathcal{O}_{\mathbf{P}_Z^n})$ is an isomorphism of graded $\mathcal{O}_Z[x_0, \dots, x_n]$ -modules,
- $H^i(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(d)) = 0$ for $0 < i < n$ and all d ,
- $H^n(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(-n-1)) \cong \mathcal{O}_Z$, and

- the natural map $H^0(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(d)) \times H^n(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(-d-n-1)) \rightarrow \mathcal{O}_Z$ is a perfect pairing of finitely generated free \mathcal{O}_Z -modules.

Remark 2.3. As the first statement indicates, $H^0(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(d))$ can be interpreted as degree d homogenous polynomials. So in light of the last statement,

$$\Gamma_*(\mathcal{O}_{\mathbf{P}_Z^n})^\vee = \bigoplus_d H^n(\mathbf{P}_Z^n, \mathcal{O}_{\mathbf{P}_Z^n}(-d-n-1))$$

should be thought of as the coalgebra dual to $\Gamma_*(\mathcal{O}_{\mathbf{P}_Z^n})$.

2.2 Relatively Flat Sheaves

We have here some standard results on relatively flat sheaves. These are at the core of many constructions in the theory of moduli schemes.

The notion of relative flatness is largely motivated by interest in studying subsheaves \mathcal{A} of a sheaf \mathcal{B} . For \mathcal{A} to remain a subsheaf of \mathcal{B} after base change, the map $\mathcal{A} \rightarrow \mathcal{B}$ must be a **universal inclusion** (also called a “universal injection”).

We find that the relationship between “subobject” and “universal inclusion” is made clear by considering when the subobject under consideration is or isn’t a sheaf: If the inclusion is not universal, then the subobject of \mathcal{B} defined as the image of \mathcal{A} is not a sheaf. Conversely, if this subobject is a sheaf, then the inclusion is universal and the sheaf in question is \mathcal{A} .

In general, it is difficult to recognize a universal inclusion. However, if the cokernel of the inclusion is relatively flat, then the map is automatically a universal inclusion. These give a class of universal inclusions we call **cokernel-flat**. If the ambient sheaf is the structure sheaf of a Z -scheme, and thus the subsheaf is an ideal, then all universal inclusions are cokernel-flat. Otherwise, one must “work” to know if a given map is a universal inclusion.

Proposition 2.4. *A sheaf \mathcal{F} on a projective Z -scheme $\pi: B \rightarrow Z$ is **relatively flat** if and only if any of the following equivalent conditions hold:*

- for all $b \in B$, the stalk \mathcal{F}_b is a flat $\mathcal{O}_{Z, \pi(b)}$ -module;
- for any affine subsets $U \subseteq B$ and $V \subseteq Z$ such that $\pi(U) \subseteq V$, we have $\mathcal{F}(U)$ is a flat $\mathcal{O}_Z(V)$ -module;
- $\Gamma_{\geq m}(\mathcal{F})$ is Z -flat for some m .

Proof. Omitted □

Proposition 2.5. *If*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is an exact sequence of quasi-coherent sheaves on a Z -scheme B , and \mathcal{H} and either \mathcal{G} or \mathcal{F} are relatively flat, then all three sheaves are.

Proof. Omitted. □

Proposition 2.6. *If \mathcal{I} is a quasi-coherent ideal sheaf on a Z -scheme B , then $\mathcal{I} \rightarrow \mathcal{O}_B$ is a universal inclusion if and only if $\mathcal{O}_B/\mathcal{I}$ is relatively flat.*

Proof. Omitted. □

Semicontinuity Theorem. ([Gro63] see also [Har77, Theorem III.12.8]) Let \mathcal{F} be a coherent Z -flat sheaf on a projective Z -scheme B . The function

$$h^i(q, \mathcal{F}) = \dim_{\mathbf{k}(q)} H^i(B_q, \mathcal{F}|_q)$$

is upper semicontinuous.

Cohomology and Base Change. ([Gro63] see also [Har77, Theorem III.12.11]) Let \mathcal{F} be a coherent Z -flat sheaf on a projective Z -scheme $\pi: B \rightarrow Z$. For $q \in Z$, if

$$R^i \pi_*(\mathcal{F}) \otimes_{\mathcal{O}_Z} \mathbf{k}(q) \rightarrow H^i(B_q, \mathcal{F}|_q)$$

is surjective, then it is isomorphism. In this case, it is an isomorphism for all q' in a open set about q , and the following statements are equivalent:

- $R^i \pi_*(\mathcal{F})$ is flat at q ;
- The restriction map

$$R^{i-1} \pi_*(\mathcal{F}) \otimes_{\mathcal{O}_Z} \mathbf{k}(q) \rightarrow H^{i-1}(B|_q, \mathcal{F}|_q)$$

is surjective.

Remark 2.7. When $R^i \pi_*(\mathcal{F}) \otimes_{\mathcal{O}_Z} \mathbf{k}(q) = H^i(B_q, \mathcal{F}|_q)$ we say *cohomology commutes with base change in degree i* . In this case, statements about $R^i \pi_*(\mathcal{F})$ are often reduced to statements about $H^i(B_q, \mathcal{F}|_q)$ (via Nakayama's lemma).

The vanishing of the first cohomology of a sheaf on a single fiber has significant implications.

Corollary 2.8. *Let \mathcal{F} be a coherent Z -flat sheaf on a projective Z -scheme $\pi: B \rightarrow Z$. If for $q \in Z$ we have $H^1(B_q, \mathcal{F}|_q) = 0$, then for all q' in a neighborhood of q the sheaf $\pi_* \mathcal{F}$ is flat at q' and $\pi_* \mathcal{F}|_{q'} = H^0(B_{q'}, \mathcal{F}|_{q'})$.*

Proof. The Semicontinuity theorem implies that for all q' in a neighborhood U of q we have $H^1(B_{q'}, \mathcal{F}|_{q'}) = 0$, and so Cohomology and Base Change for $i = 1$ gives

$$R^1\pi_*\mathcal{F}|_{q'} = H^1(B_{q'}, \mathcal{F}|_{q'}) = 0$$

in U . In particular $R^1\pi_*\mathcal{F}$ is flat on U . So again with $i = 1$, Cohomology and Base Change gives $\pi_*\mathcal{F}|_{q'} = H^0(B_{q'}, \mathcal{F}|_{q'})$ for all q' in U . Since $H^{-1}(B_{q'}, \mathcal{F}|_{q'}) = 0$ the restriction map is surjective, so Cohomology and Base Change for $i = 0$ implies $\pi_*\mathcal{F}$ is flat on U . \square

Mumford's notion of **regularity** of a sheaf leads to a practical means of knowing when one can apply the Cohomology and Base Change theorem. In the case of quasi-coherent sheaves of ideals, this concept along with the **Gotzmann regularity theorem** give powerful tools.

Definition 2.9. ([Mum66]) Let \mathbf{k} be a field. A coherent sheaf \mathcal{F} over $\mathbf{P}_{\mathbf{k}}^n$ is said to be m -regular if

$$H^i(\mathbf{P}_{\mathbf{k}}^n, \mathcal{F}(m-i)) = 0$$

for each $i > 0$.

Remark 2.10. If one puts $H^i(\mathbf{P}_{\mathbf{k}}^n, \mathcal{F}(j))$ at the location (j, i) in the plane, then the non-zero locations lie either on the x -axis, or below the line $x + y = m$. Also note, that if \mathcal{F} is extended to a \mathbb{Z} -flat family on $\mathbf{P}_{\mathbb{Z}}^n$, for some \mathbb{Z} , then the Cohomology and Base Change theorem can be applied to $\mathcal{F}(d)$ for $d \geq m$ as in Corollary 2.8.

Theorem of Castelnuovo and Mumford. ([Mum66]) Let \mathcal{F} be an m -regular coherent sheaf on $\mathbf{P}_{\mathbf{k}}^n$. Then

- $\Gamma_{\geq m}(\mathcal{F})$ is generated in degree m as a $\Gamma_*(\mathcal{O}_{\mathbf{P}_{\mathbf{k}}^n})$ -module,
- $H^i(\mathbf{P}_{\mathbf{k}}^n, \mathcal{F}(d)) = 0$ whenever $d \geq m - i$, and
- each $\mathcal{F}(d)$ for $d \geq m$ is generated by its global sections.

Corollary 2.11. *Let \mathbb{Z} be an affine Noetherian scheme. If \mathcal{F} is a \mathbb{Z} -flat coherent sheaf on $\mathbf{P}_{\mathbb{Z}}^n$ and $\mathcal{F}|_q$ is m -regular for all q in \mathbb{Z} , then*

- $\Gamma_{\geq m}(\mathcal{F})$ is \mathbb{Z} -flat and generated in degree m as a $\Gamma_*(\mathcal{O}_{\mathbf{P}_{\mathbb{Z}}^n})$ -module,
- $R^i\pi_*(\mathcal{F}(d)) = 0$ whenever $d \geq m - i$, and
- each $\mathcal{F}(d)$ for $d \geq m$ is generated by its global sections.

Proof. The second statement follows from Cohomology and Base change.

The last can be checked considering $b \in \mathbf{P}_Z^n$. Write $q = \pi(b)$. Cohomology and Base change gives the surjection $\Gamma(\mathbf{P}_Z^n, \mathcal{F}(d)) \rightarrow \Gamma(\mathbf{P}_{\mathbf{k}(q)}^n, \mathcal{F}(d)|_q)$, and the theorem of Castelnuovo and Mumford gives the surjection

$$\Gamma(\mathbf{P}_{\mathbf{k}(q)}^n, \mathcal{F}(d)|_q) \otimes_{\mathcal{O}_{\mathbf{P}_{\mathbf{k}(q)}^n}} \mathbf{k}(b) \rightarrow (\mathcal{F}(d)|_q)|_b.$$

The $\mathbf{k}(b)$ -vector spaces $(\mathcal{F}(d)|_q)|_b$ and $\mathcal{F}(d)|_b$ equal, so we get a surjection $\Gamma(\mathbf{P}_Z^n, \mathcal{F}(d)) \otimes_{\mathcal{O}_{\mathbf{P}_Z^n}} \mathbf{k}(b) \rightarrow \mathcal{F}(d)|_b$, and can apply Nakayama's lemma.

The first statement requires consideration of the sheaves $\mathcal{O}(d)$, $\mathcal{F}(m)$ and $\mathcal{F}(d+m)$. Cohomology commutes with base change for all these sheaves, so we can consider the question on the fiber. The last fact we need is that $(\pi_* \mathcal{O}(d) \otimes_{\mathcal{O}_Z} \pi_* \mathcal{F}(m))|_q \rightarrow (\pi_* \mathcal{F}(d+m))|_q$ factors through the epimorphism

$$(\pi_* \mathcal{O}(d) \otimes_{\mathcal{O}_Z} \pi_* \mathcal{F}(m))|_q \rightarrow \pi_* \mathcal{O}(d)|_q \otimes_{\mathbf{k}(q)} \pi_* \mathcal{F}(m)|_q.$$

We can now appeal to the theorem of Castelnuovo and Mumford and apply Nakayama's lemma. \square

Gotzmann Regularity. ([Got78] see also [BH93, Theorem 4.3.2]) If L is a closed subscheme of $\mathbf{P}_{\mathbf{k}}^n$ with Hilbert polynomial $p_{\mathcal{O}_L}(t)$, then there is a unique expansion

$$p_{\mathcal{O}_L}(t) = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \cdots + \binom{t+a_m-(m-1)}{a_m}$$

for weakly decreasing integers $a_1 \geq a_2 \geq \cdots \geq a_m$. Furthermore, for m in the above expansion \mathcal{I}_L , is m -regular. The integer m is called the **Gotzmann number** of $p_{\mathcal{O}_L}$.

Remark 2.12. It is interesting to note that the Gotzmann number depends only on the polynomial. Not even on the dimension of the ambient projective space.

Corollary 2.13. Consider a B -flat closed subscheme L of \mathbf{P}_B^n with Hilbert polynomial $p_{\mathcal{O}_L}$ and Gotzmann number m . The sheaves $\pi_* \mathcal{I}_L$ and $\pi_* \mathcal{O}_L$ are m -regular, and

$$0 \rightarrow \pi_* \mathcal{I}_L(d) \rightarrow \pi_* \mathcal{O}_{\mathbf{P}_B^n}(d) \rightarrow \pi_* \mathcal{O}_L(d) \rightarrow 0$$

is an exact sequence of Z -flat sheaves.

Proof. \mathcal{O}_L is B -flat, so after pull back to $\mathbf{P}_{\mathbf{k}(q)}^n$ the sequence

$$0 \rightarrow \mathcal{I}_L|_q \rightarrow \mathcal{O}_{\mathbf{P}_{\mathbf{k}(q)}^n} \rightarrow \mathcal{O}_L|_q \rightarrow 0$$

is exact. Here Gotzmann regularity implies $H^i(\mathbf{P}_{\mathbf{k}(q)}^n, \mathcal{I}_L|_q(d)) = 0$ for all $i > 0$. These groups can be computed by the same Čech-complex as the $R^i \pi_*(\mathcal{I}_L(d) \otimes_{\mathcal{O}_{\mathbf{P}_B^n}} \mathbf{k}(q))$'s, and are thus the same. So we may apply Cohomology and Base Change to conclude $R^i \pi_*(\mathcal{I}_L(d)) = 0$ for $i > 0$. Consequently, both $\pi_* \mathcal{I}_L(d)$ and $\pi_* \mathcal{O}_L(d)$ are flat. \square

Notation. When we have a projective Z -scheme B and a sheaf \mathcal{F} on B , we will say some version of the statement

“The Hilbert polynomial of \mathcal{F} is independent of Z .”

to indicate that there is a fixed polynomial that equals the Hilbert polynomial of $\mathcal{F}|_q$ regardless of the choice of point $q \in Z$.

Later, we will need to generalize the following theorem to reduced schemes.

Hilbert Polynomials and Relative Flatness. ([Har77, proof of Theorem III.9.9]) If Z is an integral Noetherian scheme. Let \mathcal{F} be a coherent sheaf on a projective Z -scheme B . Then \mathcal{F} is Z -flat if and only if Hilbert polynomial of $\mathcal{F}|_q$ is independent of $q \in Z$.

2.3 Representability of certain functors

We will use certain schemes in a way that makes it convenient to think of them in terms of the functors they represent. Specifically, **flattening stratifications**, **Quot schemes**, **Hilbert schemes**, and **scheme theoretic images**.

Existence of the Universal Flattening Stratification. ([Gro61]) If $B \rightarrow Z$ is projective and \mathcal{F} is a coherent \mathcal{O}_B module, then there is Z -scheme $Z_{\text{flat}}^{\mathcal{F}} \rightarrow Z$ which represents the functor

$$Z' \mapsto \{h: Z' \rightarrow Z \mid h^* \mathcal{F} \text{ is } Z'\text{-flat}\}.$$

Furthermore, $Z_{\text{flat}}^{\mathcal{F}}$ is a disjoint union of locally closed subschemes of Z called **strata**, one of which is topologically open and dense in Z .

Notation. If \mathcal{F} is a sheaf over a projective Z -scheme and $p_{\mathcal{F}} = p_{\mathcal{F}}(t)$ is a polynomial in t , we write $Z_{p_{\mathcal{F}}}$ for the disjoint union of locally closed subschemes over which \mathcal{F} is flat and has Hilbert polynomial $p_{\mathcal{F}}$. This is potentially confusing since suggests that $p_{\mathcal{F}}$ depends on \mathcal{F} . However, this notation should simply indicate that we are introducing a polynomial $p_{\mathcal{F}}$ that we wish to associate with the sheaf \mathcal{F} .

Representability of the Quot Functor. ([Gro61]) Given a coherent sheaf \mathcal{F} over a projective Z -scheme B the functor

$$Z' \mapsto \{Z'\text{-flat quotients } \mathcal{G} \text{ of } \pi_B^* \mathcal{F} \text{ on } Z' \times_Z B\}$$

is representable by a projective B -scheme $\text{Quot}(\mathcal{F}/B/Z)$.

Remark 2.14. From this we have the Hilbert scheme which is $\text{Hilb}(B/Z) = \text{Quot}(\mathcal{O}_B/B/Z)$.

Scheme Theoretic Image. ([Sta13, Tag 01R5]) Given a morphism of schemes $\phi: V \rightarrow W$. There exists a closed subscheme $\overline{\phi(V)} \subseteq W$ called the **scheme theoretic image** such that ϕ factors through $\overline{\phi(V)}$ and $\overline{\phi(V)}$ is initial among such closed subschemes of W .

3 The Moduli of Quasi-Splines Sheaves

In this section, we construct in Theorem 3.8 the moduli of cokernel-flat families of quasi-spline sheaves CFQS. The functor represented by CFQS is

$$\mathcal{CFQS}(Z) = \{\mathcal{S} \in \mathcal{QS}(Z) \mid \mathcal{G} = \text{cok}(\mathcal{S} \rightarrow \mathcal{O}_{Z \times_T Y}^S) \text{ is } Z\text{-flat}\}.$$

Where \mathcal{QS} is the functor of families of quasi-spline schemes from the introduction.

When Y is T -flat we have Theorem 3.10 which states the existence of the scheme $QS = CFQS$ representing \mathcal{QS} . This is based on Lemma 3.9 makes the observation that a quasi-spline sheaf \mathcal{S} over projective, flat Z -scheme B is a Z -family if and only if the cokernel \mathcal{G} of the inclusion $\mathcal{S} \rightarrow \mathcal{O}_B^S$ is Z -flat.

Definition 3.1. A sheaf of quasi-splines over a scheme B is a quasi-coherent \mathcal{O}_B -subalgebra of \mathcal{O}_B^S .

Definition 3.2. A Z -family of quasi-spline sheaves over a Z -scheme B to be

- a sheaf of quasi-splines \mathcal{S} over a B such that
- for any morphism $f: Z' \rightarrow Z$, the pullback $\pi_B^* \mathcal{S}$ is a sheaf of quasi-splines over $Z' \times_Z B$.

Definition 3.3. We say that a Z -family \mathcal{S} of quasi-spline sheaves over B is **cokernel-flat** if the sheaf \mathcal{G} is the exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_B^S \rightarrow \mathcal{G} \rightarrow 0$$

is Z -flat.

Lemma 3.4. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of coherent sheaves over a projective Z -scheme B . If \mathcal{G} is Z -flat, then the functor $Z' \mapsto \{h \in \text{Mor}_T(Z', Z) \mid h^* \phi = 0\}$ is representable by a closed subscheme $V(\phi) \subseteq Z$.

Proof. It suffices to work locally on Z and assume that $B \subseteq \mathbf{P}_Z^n$. Provided d is sufficiently large, $\mathcal{F}(d)$ is generated by global sections and $\Gamma(B, \mathcal{G}(d))$ is a flat \mathcal{O}_Z -module. Consider the image under $\phi(d): \Gamma(B, \mathcal{F}(d)) \rightarrow \Gamma(B, \mathcal{G}(d))$ of generators $\{f_i\}_i \subseteq \Gamma(B, \mathcal{F}(d))$. Since Z is local, $\Gamma(B, \mathcal{G}(d))$ is free and we can choose a basis $\{g_j\}_j$. For each f_i we have an expansion

$$\phi(f_i) = \sum_j c_{ij} g_j.$$

The condition that $\phi = 0$ is the same as $c_{ij} = 0$ for all ij . So we set $V(\phi) = V(\{c_{ij}\})$. After any base change, $\mathcal{F}(d)$ is still generated by the f_i 's and the cohomology and base change theorem implies that the g_j 's remain linearly independent. So the vanishing of ϕ is exactly the condition that the c_{ij} 's vanish. \square

Definition 3.5. Fix a scheme B and consider a quasi-coherent subsheaf \mathcal{S} of \mathcal{O}_B^s . Write $\gamma: \mathcal{O}_B^s \rightarrow \mathcal{G}$ for the cokernel of the inclusion $\iota: \mathcal{S} \rightarrow \mathcal{O}_B^s$. Write $\delta: \mathcal{O}_B \rightarrow \mathcal{O}_B^s$ for the diagonal inclusion and $\mu: \mathcal{O}_B^s \otimes_{\mathcal{O}_B} \mathcal{O}_B \rightarrow \mathcal{O}_B^s$ for entry-wise multiplication. We define

- $\kappa: \mathcal{O}_B \rightarrow \mathcal{G}$ to be the composition $\gamma \circ \delta$, and
- $m: \mathcal{S} \otimes_{\mathcal{O}} \mathcal{S} \rightarrow \mathcal{G}$ to be $\gamma \circ \mu \circ (\iota \otimes \iota)$.

Lemma 3.6. *A quasi-coherent subsheaf \mathcal{S} of \mathcal{O}_B^s is a quasi-spline sheaf if and only if*

$$\kappa = 0 \text{ and } m = 0.$$

for the maps κ and m of Definition 3.5.

Proof. $\kappa = 0$ and $m = 0$ if and only if they factor through the kernel of γ . So both δ and $\mu \circ (\iota \otimes \iota)$ factor through \mathcal{S} . This means precisely that $\mathcal{O}_B \subseteq \mathcal{S}$ and \mathcal{S} is closed under entry-wise multiplication in \mathcal{O}_B^s . \square

Definition 3.7. Consider the Quot scheme $Q = \text{Quot}(\mathcal{O}_Y^s/Y/T)$ and the map

$$\phi = \kappa \oplus m: \mathcal{O}_{Q \times_T Y} \oplus (\mathcal{S} \otimes_{\mathcal{O}_{Q \times_T Y}} \mathcal{S}) \rightarrow \mathcal{G} \oplus \mathcal{G}.$$

where κ and m are the maps from Definition 3.5 for the universal kernel $\mathcal{S} \subseteq \mathcal{O}_{Q \times_T Y}^s$. We set

$$\text{CFQS} = \mathbf{V}(\phi) \subseteq Q$$

as in Lemma 3.4.

Theorem 3.8. *The functor \mathcal{CFQS} is represented by CFQS.*

Proof. For any such family we know by Lemma 3.9 that the cokernel \mathcal{G} of the inclusion $\mathcal{S} \rightarrow \mathcal{O}_{Z \times_T Y}^s$ is Z -flat. Furthermore, the quasi-spline sheaf is determined by the map $\mathcal{O}_{Z \times_T Y}^s \rightarrow \mathcal{G}$. This means that there is a natural transformation from this functor into the Quot scheme.

The identification

$$\mathcal{S} \otimes_{\mathcal{O}_{Q \times_T Y}} \mathcal{S} \otimes_{\mathcal{O}_{Q \times_T Y}} \mathcal{O}_{Z \times_T Y} \cong (\mathcal{S} \otimes_{\mathcal{O}_{Q \times_T Y}} \mathcal{O}_{Z \times_T Y}) \otimes_{\mathcal{O}_{Z \times_T Y}} (\mathcal{S} \otimes_{\mathcal{O}_{Q \times_T Y}} \mathcal{O}_{Z \times_T Y})$$

shows that the κ and m maps of Definition 3.5 over $Q \times_T Y$ pull back to the κ and m maps over $Z \times T$. For any quasi-spline sheaf over $Z \times T$ these maps vanish by Lemma 3.6, so the morphism factors through CFQS by Lemma 3.4.

On the other hand, the universal kernel \mathcal{S} restricted to CFQS is a quasi-spline sheaf, again by Lemma 3.6. Lemma 3.9 guarantees this sheaf is a CFQS-family. Consequently, points in $\text{CFQS}(Z)$ produce distinct Z -families of quasi-spline sheaves over $Z \times_T Y$, and so the natural transformation is a bijection. \square

Lemma 3.9. *Given a flat, projective Z -scheme B , a quasi-spline sheaf \mathcal{S} is a Z -family over B if and only if the cokernel \mathcal{G} of the inclusion $\mathcal{S} \rightarrow \mathcal{O}_B^s$ is Z -flat. Thus if Y is flat over T , then $\mathcal{QS} = \mathcal{CFQS}$.*

Proof. B is flat, thus so is \mathcal{O}_B^s . Consider the exact sequence at a point $b \in B$:

$$0 \rightarrow \mathcal{S}_b \rightarrow \mathcal{O}_{B,b}^s \rightarrow \mathcal{G}_b \rightarrow 0.$$

Denote the image of b in Z by q . Tensoring with the sheaf $\mathbf{k}(q)$ we get the Tor exact sequence

$$0 \rightarrow \mathrm{Tor}_1^{\mathcal{O}_Z}(\mathcal{G}_b, \mathbf{k}(q)) \rightarrow \mathcal{S}_b \otimes_{\mathcal{O}_Z} \mathbf{k}(q) \rightarrow \mathcal{O}_{B,b}^s \otimes_{\mathcal{O}_Z} \mathbf{k}(q) \rightarrow \mathcal{G}_b \otimes_{\mathcal{O}_Z} \mathbf{k}(q) \rightarrow 0.$$

So we see that $\mathcal{S} \rightarrow \mathcal{O}_B^s$ is a universal inclusion if and only if $\mathrm{Tor}_1^{\mathcal{O}_Z}(\mathcal{G}_b, \mathbf{k}(q)) = 0$ for all $b \in B$, i.e. \mathcal{G} is Z -flat. \square

Theorem 3.10. *If Y is T -flat, $QS = CFQS$ represents \mathcal{QS} .*

Proof. Combine Lemma 3.9. and Theorem 3.8. \square

Remark 3.11. If Y is not T -flat, there is no chance that $QS = CFQS$ (in the way presented here). For instance, one can take $\mathcal{S} =$ the diagonal copy of $\mathcal{O}_Y \subseteq \mathcal{O}_Y^s$. The inclusion is universal and it splits \mathcal{O}_Y^s into a direct sum $\mathcal{O}_Y \oplus \mathcal{O}_Y^{s-1}$. So the cokernel is isomorphic to \mathcal{O}_Y^{s-1} , and not T -flat.

4 The Moduli of Ideal Difference-Conditions

We begin by constructing the moduli of ideal difference-conditions C in Theorem 4.4 as a flattening stratification of a certain sheaf over $\mathrm{Hilb}(Y/T)^{(2)}$. To produce our “compactification” of this scheme, we show in Proposition 4.9 it is a subscheme of $CFQS \times_T \mathrm{Hilb}(Y/T)^{(2)}$, and define the compactification to be the scheme theoretic image \overline{C} of the inclusion. Finally, we argue via Proposition 4.10 that this compactification is the “correct” one.

4.1 The Moduli of Ideal Difference-Conditions

For ideal difference-conditions defined by a collection of ideals $(\mathfrak{I}_{jk})_{jk}$ over a Z -scheme B , we consider the morphism

$$\Delta: \mathcal{O}_B^s \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk}. \quad (3)$$

which sends $(g_1, \dots, g_s) \mapsto (g_j - g_k + \mathfrak{I}_{jk})_{jk}$. Not all collections of ideals are well behaved.

Definition 4.1. Recall the notion of a **Z-family** of ideal difference-conditions: Under base change along any morphism $Z' \rightarrow Z$ the sequence

$$0 \rightarrow \mathcal{S}_{\mathfrak{I}} \rightarrow \mathcal{O}_B^s \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk}.$$

remains exact, and for each jk the sequence

$$0 \rightarrow \mathfrak{I}_{jk} \rightarrow \mathcal{O}_B \rightarrow \mathcal{O}_B / \mathfrak{I}_{jk} \rightarrow 0$$

also remains exact. This means sheaves $(\mathcal{S}_{\mathfrak{I}}, (\mathfrak{I}_{jk})_{jk})$ “remain themselves.” after such a change of base.

Lemma 4.2. *A collection of ideals $(\mathfrak{I}_{jk})_{jk}$ over a Z -scheme B define a Z -family of ideal difference-conditions if and only if both $\bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk}$ and the cokernel of*

$$\Delta: \mathcal{O}_B^s \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk}$$

are Z -flat. In this case, the cokernel \mathcal{G} of $\mathcal{S}_{\mathfrak{I}} \rightarrow \mathcal{O}_B^s$ is automatically Z -flat.

Proof. For ideals, universal inclusions are equivalent to flatness of their cokernels, so a Z -family requires $\mathcal{O}_{Z \times_T Y} / \mathfrak{I}_{jk}$ is Z -flat for all jk . Given this, the additional required conditions reveal themselves after considering the two standard exact sequences associated to the morphism in Equation (3):

$$0 \rightarrow \mathcal{S}_{\mathfrak{I}} \rightarrow \mathcal{O}_B^s \rightarrow \mathcal{G} \rightarrow 0 \tag{4}$$

and

$$0 \rightarrow \mathcal{G} \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk} \rightarrow \mathcal{H} \rightarrow 0. \tag{5}$$

If either of these exact sequences fail to be exact after base change, $\mathcal{S}_{\mathfrak{I}}$ will no longer be the kernel of Equation (3).

Since $\bigoplus_{jk} \mathcal{O}_{Z \times_T Y} / \mathfrak{I}_{jk}$ must be Z -flat, universal exactness of the second standard sequence is equivalent to the Z -flatness of \mathcal{H} . This implies the Z -flatness of \mathcal{G} , and thus the exactness of the first standard sequence. \square

Definition 4.3. Given a projective scheme Y/T , denote the structure sheaf of product $\text{Hilb}(Y/T)^{(s)}$ of Hilbert Schemes by \mathcal{O} . Over this product we have the morphism

$$\Delta: \mathcal{O}^s \rightarrow \bigoplus_{jk} \mathcal{O} / \mathfrak{I}_{jk}.$$

Denote the cokernel by \mathcal{H} and we define **the moduli of ideal difference-conditions** $C =$ the universal flattening stratification for \mathcal{H} .

Theorem 4.4. *The functor*

$$\mathcal{C}(Z) = \{Z\text{-families of ideal difference conditions on } Z \times_T Y\}$$

is representable by C.

Proof. The definition of the Hilbert Scheme, the universal flattening stratification, and Lemma 4.2 give the result. \square

4.2 Compatible Pairs and Compactification of the Moduli of Ideal Difference-Conditions

We now have a construction of the moduli scheme of ideal difference conditions. However, to construct a satisfying “compactification,” we present it in a slightly different way. This involves the observation that the assignment

$$(\mathcal{I}_{jk})_{jk} \mapsto \mathcal{S}_{\mathcal{I}}$$

defines a morphism $C \rightarrow \text{CFQS}$. It will turn out that the resulting morphism

$$C \rightarrow \text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)}$$

is an inclusion of C as a locally closed subscheme, and its scheme theoretic closure \overline{C} is the “correct” compactification. The correctness of \overline{C} is based on the existence of a universal family of compatible pairs (Definition 4.5) and its universality (Proposition 4.10).

Definition 4.5. A pair of a quasi-spline sheaf \mathcal{S} and a $\binom{s}{2}$ -tuple of ideal sheaves $(\mathcal{I}_{jk})_{jk}$ over a scheme B is called a **compatible pair** if the composition

$$\mathcal{S} \rightarrow \mathcal{O}^s \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathcal{I}_{jk}$$

is zero.

Definition 4.6. Denote by \mathcal{O} the structure sheaf of $\text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)} \times_T Y$. Over $\text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)} \times_T Y$ we have the universal pair $(\mathcal{S}, (\mathcal{I}_{jk})_{jk})$ and the compatibility map

$$\psi: \mathcal{S} \rightarrow \bigoplus_{jk} \mathcal{O} / \mathcal{I}_{jk}.$$

The sheaf $\bigoplus_{jk} \mathcal{O} / \mathcal{I}_{jk}$ is relatively flat over $\text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)}$, so Lemma 3.4 produces the **moduli of compatible pairs**

$$P = V(\psi) \subseteq \text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)}.$$

There is a natural morphism $f: C \rightarrow P$ which sends $(\mathfrak{I}_{jk})_{jk} \mapsto (\mathcal{S}_{\mathfrak{I}}, (\mathfrak{I}_{jk})_{jk})$.

Lemma 4.7. *Let Z be a locally Noetherian and \mathcal{F} be a coherent sheaf over a reduced projective Z -scheme B . Then \mathcal{F} is Z -flat if and only if the Hilbert polynomial of \mathcal{F} is locally independent of Z .*

Proof. The question is local on Z , so assume Z is affine and thus has finitely many irreducible components. Write Z_1, \dots, Z_k for the irreducible components of Z , and write $Z_{\text{flat}}^{\mathcal{F}}$ for the flattening stratification of Z for \mathcal{F} . Over each Z_i the restriction \mathcal{F} is flat by Hartshorne III.9.9, so we get a morphism $\phi_i: Z_i \rightarrow Z_{\text{flat}}^{\mathcal{F}}$. ϕ_i and ϕ_j agree on $Z_i \cap Z_j$. It is a quick check (the Chinese Remainder Theorem) to see that one can patch maps on a pair of closed subschemes to produce one on their union if the maps agree on the intersection. So these maps define $\phi_{ij}: Z_i \cup Z_j \rightarrow Z_{\text{flat}}^{\mathcal{F}}$. If we take this as a base case, the same argument produces a map $\phi_{i_1 \dots i_\ell}: Z_{i_1} \cup \dots \cup Z_{i_\ell} \rightarrow Z_{\text{flat}}^{\mathcal{F}}$ from $\phi_{i_1 \dots i_{\ell-1}}$ and ϕ_{i_ℓ} . There are only finitely many components, so we get a morphism $\phi: Z \rightarrow Z_{\text{flat}}^{\mathcal{F}}$. This morphism is a section of the map $Z_{\text{flat}}^{\mathcal{F}} \rightarrow Z$. Since Z is reduced and $Z_{\text{flat}}^{\mathcal{F}}$ is a locally closed subscheme of Z , this map is an isomorphism. \square

Definition 4.8. Denote the **compactification by compatible pairs** \overline{C} to be the scheme theoretic image of C in P .

Proposition 4.9. $C \rightarrow \overline{C}$ is a locally closed immersion.

Proof. Since \overline{C} is a closed subset of P , we will show that $f: C \rightarrow P$ is a locally closed immersion. The question is local on T , so assume $Y \subseteq \mathbb{P}_T^n$. This allows us to talk about Hilbert polynomials for Z -flat sheaves on schemes of the form $Z \times_T Y$.

First we establish that if we fix polynomials $p_{\mathcal{G}}$ and $p_{\mathfrak{I}_{jk}}$ for each jk , the scheme $C_{p_{\mathcal{G}}, p_{\mathfrak{I}}}$ is a union of connected components of C . To do this, we must verify that the Hilbert polynomials of \mathcal{G} and the \mathfrak{I}_{jk} 's are locally independent of the base.

The Hilbert polynomials $p_{\mathfrak{I}_{jk}}(t)$ for each jk are locally independent of C by virtue of the fact that the \mathfrak{I}_{jk} 's pull back from $\text{Hilb}(Y/T)^{(s)}$. For the Hilbert polynomial of \mathcal{G} , observe that we have the equation

$$p_{\mathcal{G}}(t) = \binom{s}{2} p_{\mathcal{O}_Y}(t) - \sum_{jk} p_{\mathfrak{I}_{jk}}(t) - p_{\mathcal{H}}(t). \quad (6)$$

where we continue to denote the cokernel of the morphism $\mathcal{G} \rightarrow \bigoplus_{jk} \mathcal{O}_B / \mathfrak{I}_{jk}$ by \mathcal{H} . This equation shows that the local constancy of $p_{\mathcal{H}}(t)$ is equivalent to the local constancy of $p_{\mathcal{G}}(t)$ (provided the $p_{\mathfrak{I}_{jk}}(t)$'s are locally independent of the base). As the flattening stratification of \mathcal{H} , $p_{\mathcal{H}}(t)$ is locally independent of C .

Before considering P , we establish the map $C_{p_{\mathcal{G}}, p_{\mathfrak{I}}} \rightarrow \text{Hilb}(Y/T)^{(s)}$ is a locally closed immersion. This is topological statement since we know that, as a flattening stratification,

C is the disjoint union of locally closed subschemes. Consider the image Λ of $C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$ as a topological space. Choose a connected component Λ' of Λ . The Hilbert polynomials of \mathcal{H} and the ideals \mathcal{I}_{jk} are locally independent of Λ' , so there is an open subset U of the closure $\overline{\Lambda'}$ in $\text{Hilb}(Y/T)^{(s)}_{(2)}$ on which all these polynomials are locally independent of the point in U . This is a constructible set containing Λ' . Equipped with its reduced scheme structure, the local independence of the Hilbert polynomials of \mathcal{H} and the \mathcal{I}_{jk} 's imply by Lemma 4.2 these sheaves are flat over U . So it admits a section $U \rightarrow C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$ and we can conclude that $U = \Lambda'$. In other words, any connected component Λ' of Λ is the homeomorphic image of a connected component of $C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$. Thus $C_{p_{\mathcal{G}}, p_{\mathcal{I}}} \rightarrow \text{Hilb}(Y/T)^{(s)}_{(2)}$ is a locally closed immersion.

Finally, we consider the morphism $C \rightarrow P$. Denote by $\text{CFQS}_{p_{\mathcal{G}}}$ the component of CFQS over which \mathcal{G} has Hilbert polynomial $p_{\mathcal{G}}$. We see that $C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$ is carried to $\text{CFQS}_{p_{\mathcal{G}}} \times_T C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$. This is a locally closed subscheme of $\text{CFQS} \times_T \text{Hilb}(Y/T)^{(s)}_{(2)}$, so $P \cap (\text{CFQS}_{p_{\mathcal{G}}} \times_T C_{p_{\mathcal{G}}, p_{\mathcal{I}}})$ is a locally closed subscheme of P . $C_{p_{\mathcal{G}}, p_{\mathcal{I}}}$ itself is identified with the open subset of $P \cap (\text{CFQS}_{p_{\mathcal{G}}} \times_T C_{p_{\mathcal{G}}, p_{\mathcal{I}}})$ over which $\mathcal{S} \rightarrow \mathcal{S}_{\mathcal{I}}$ is an isomorphism. To be sure that this is an open set, apply Γ_* to the map, and notice this set coincides with the points where the cokernel and kernel vanish. Thus as an open subscheme of a locally closed subscheme of P , it is locally closed. \square

We conclude with the universal property of \overline{C} .

Proposition 4.10. *Let $C \rightarrow H$ be a morphism to a scheme such that $H \times_T Y$ is equipped with an H -family of compatible pairs whose restriction to $C \times_T Y$ is the universal family of ideal difference-conditions. Assume H equals the scheme theoretic image of C in H . Then the morphism $H \rightarrow P$ factors through \overline{C} .*

Proof. $H \times_P \overline{C}$ equals H because it is a closed subscheme of H containing the image of C :

$$\begin{array}{ccccc} C & \longrightarrow & H \times_P \overline{C} & \longrightarrow & \overline{C} \\ & \searrow & \downarrow & & \downarrow \\ & & H & \longrightarrow & P \end{array}$$

\square

4.3 Degeneracy Loci and Rank Strata

The construction of the moduli space gives a space in which one can move without changing the Hilbert polynomial. It seems likely that one would be interested in the the whole Hilbert series, not just the polynomial. To address this, we present a way in which one can stratify the moduli space by pieces on which the Hilbert series is

unchanged using degeneracy loci. This can be done provided cohomology and base change commute for $\mathcal{O}_B^s(d)$ and $\bigoplus_{jk}(\mathcal{O}_B/\mathfrak{I}_{jk})(d)$ for all d . This condition holds in the most important case of $B = \mathbf{P}^n$ as shown in Corollary 4.13.

Definition 4.11. Given a morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$, of flat coherent sheaves on a scheme Z , we say

$$\text{rank}(\phi) \leq r$$

if the induced morphism

$$\wedge^r \phi: \wedge^r \mathcal{F} \rightarrow \wedge^r \mathcal{G}$$

is zero. This map is a global section

$$\wedge^r \phi \in \Gamma(Z, \wedge^r(\mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{F}^\vee)).$$

The sheaf $\wedge^r(\mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{F}^\vee)$ is locally free, so $\wedge^r \phi$ defines a **scheme of zeros** $(\wedge^r \phi)_0$. This scheme is called the r^{th} **degeneracy locus** of ϕ , and we will denote it by $\text{DL}_r(\phi)$.

Proposition 4.12. *Let B be a projective, flat Z -scheme. Consider $\binom{s}{2}$ quasi-coherent ideals $\mathfrak{I}_{jk} \subseteq \mathcal{O}_B$ with Z -flat quotients $\mathcal{O}_B/\mathfrak{I}_{jk}$. If $H^1(B_q, \bigoplus_{jk} \mathcal{O}_{B_q}/(\mathfrak{I}_{jk})_q)$ and $H^1(B_q, \mathcal{O}_{B_q}^s)$ vanish for all $q \in Z$, then the locally closed subset on which $\pi_* \mathcal{S}(d)$ has rank ρ is*

$$\text{DL}_r(\Delta(d)) \setminus \text{DL}_{r-1}(\Delta(d))$$

where

$$\Delta(d): \pi_* \mathcal{O}_B^s(d) \rightarrow \pi_* \bigoplus_{jk} (\mathcal{O}_B/\mathfrak{I}_{jk})(d)$$

and

$$r = \text{rank}(\pi_* \mathcal{O}_B^s(d)) - \rho.$$

Proof. The vanishing of $H^1(B_q, \bigoplus_{jk} \mathcal{O}_{B_q}/(\mathfrak{I}_{jk})_q)$ and $H^1(B_q, \mathcal{O}_{B_q}^s)$ plus the Corollary 2.8 guarantee that $\pi_* \bigoplus_{jk} (\mathcal{O}_B/\mathfrak{I}_{jk})(d)$ and $\pi_* \mathcal{O}_B^s(d)$ are locally free \mathcal{O}_Z -modules.

Locally, $\Delta(d)$ is a matrix, and its cokernel is flat over a scheme $Z' \rightarrow Z$ exactly when this matrix pulls back to a constant rank matrix over Z' . This is the same as requiring that Z' maps into $\text{DL}_r(\Delta(d)) \setminus \text{DL}_{r-1}(\Delta(d))$ for some r .

The formula relating r and ρ follows from the fact that if $\Delta(d)$ is constant rank, then $\mathcal{S}(d)$ is flat of the given rank. \square

Corollary 4.13. *Set $B = \mathbf{P}_Z^n$ and $\mathfrak{I}_{jk} = \mathcal{O}_B(-D_{jk})$ for relatively effective Cartier divisors D_{jk} . If $n \neq 2$ or $\deg(D_{jk}) < n-1$ for all jk , then the locus on which $\pi_* \mathcal{S}(d)$ has rank ρ is*

$$\text{DL}_r(\Delta(d)) \setminus \text{DL}_{r-1}(\Delta(d))$$

where

$$r = s \binom{d+n}{2} - \rho.$$

Proof. The conditions on n and/or the D_{jk} 's guarantee these sheaves have no first cohomology. So we may apply Proposition 4.12. \square

Remark 4.14. To connect this with the moduli space, one might begin with the product of Hilbert schemes of degree d_{jk} hypersurfaces $\prod_{jk} \mathbf{P}(\Gamma(\mathbf{P}^n, \mathcal{O}(d_{jk}))^\vee)$ equipped with the bundle $\mathcal{H} = \text{cok} \Delta$. Then Z would be taken to be the flattening stratification of \mathcal{H} . These degeneracy loci then give the stratification of the moduli space on which the Hilbert series, not just the Hilbert polynomials are unchanged.

5 The Moduli of Quasi-Splines

In this section, we prove for a T-flat closed subscheme Y of a projective space bundle $\mathbf{P}(\mathcal{V})$, the functor which picks out a T-family of quasi-spline sheaves \mathcal{S} and a section of a d^{th} twist of \mathcal{S} is representable, provided d is sufficiently large. The bound we find for d depends on the Hilbert polynomials of \mathcal{O}_Y and \mathcal{S} . Since Y is T-flat, we have a scheme QS representing \mathcal{QS} , and this scheme equals CFQS.

The crucial thing we need is a bound for the Castelnuovo-Mumford regularity of \mathcal{S} . This is why we assume Y is a closed subscheme of a projective space bundle over Z . This will allow use to eventually use the Gotzmann regularity theorem.

Assumption. In the statements below, we will assume that B is a subscheme of a projective bundle $\mathbf{P}(\mathcal{V}) = \text{Proj}(\text{Sym}^\bullet \mathcal{V})$ over Z where \mathcal{V} is flat and finite rank on Z .

To identify \mathcal{S} with an ideal sheaf, we first introduce an auxiliary projective space K .

Definition 5.1. Set

$$K = \text{Proj} (\text{Sym}^\bullet \mathcal{V})[E_1, \dots, E_s]$$

where E -variables are in degree 1 (this is the projective closure of the product of $\mathbf{A}^{(s-1)}$ with the the affine cone over $\mathbf{P}(\mathcal{V})$).

Now we define the space N over which our ideal sheaf will live.

Definition 5.2. B can be found as a subscheme of the copy of $\mathbf{P}(\mathcal{V})$ in K , cut out by $E_1 = \dots = E_s = 0$. The first order infinitesimal neighborhood of $\mathbf{P}(\mathcal{V})$ is given by

$$N_{\mathbf{P}(\mathcal{V})} = \mathbf{V}(\langle E_1, \dots, E_s \rangle^2) \subseteq K.$$

This is a scheme over $\mathbf{P}(\mathcal{V})$, so we define

$$N = B \times_{\mathbf{P}(\mathcal{V})} N_{\mathbf{P}(\mathcal{V})} \subseteq K.$$

Definition 5.3. Consider the inclusion of $\Gamma_*(\mathcal{O}_K)$ -modules

$$(\Gamma_*(\mathcal{O}_K)/\Gamma_*(\mathcal{I}_B))^s(-1) \longrightarrow \Gamma_*(\mathcal{O}_K)/\Gamma_*(\mathcal{I}_N)$$

which sends

$$(g_1, \dots, g_s) \mapsto g_1 E_1 + \dots + g_s E_s.$$

The image of this map is $\Gamma_*(\mathcal{I}_B)/\Gamma_*(\mathcal{I}_N)$. The map induces an inclusion

$$\mathcal{O}_B(-1)^s \rightarrow \mathcal{O}_N.$$

and an isomorphism $\mathcal{O}_B(-1)^s \cong \mathcal{I}_{B \subseteq N}$. $\mathcal{S}(-1)$ is carried to an ideal of \mathcal{O}_N we denote by $\mathcal{I}_{L \subseteq N}$, and we write L for the closed subscheme of N defined by this ideal.

This way we translate questions about \mathcal{S} into questions about the ideal sheaf $\mathcal{I}_{L \subseteq N}$.

Proposition 5.4. $\pi_* \mathcal{S}(d) \cong \pi_* \mathcal{I}_{L \subseteq N}(d+1)$.

Proof. The $\mathcal{S}(-1) \rightarrow \mathcal{O}_N$ is an inclusion because $\tilde{(\cdot)}$ is exact. \square

Lemma 5.5. *If B is \mathbb{Z} -flat and \mathcal{S} is a \mathbb{Z} -family, then L and N are \mathbb{Z} -flat.*

Proof. As \mathcal{O}_B -modules $\mathcal{O}_N \cong \mathcal{O}_B \oplus \mathcal{O}_B(-1)^s$, and $\mathcal{O}_L \cong \mathcal{O}_B \oplus \mathcal{G}(-1)$. We know \mathcal{G} is \mathbb{Z} -flat by Lemma 3.9. \square

Proposition 5.6. *If B is \mathbb{Z} -flat and \mathcal{S} is a \mathbb{Z} -family, then $\pi_* \mathcal{S}(d)$ is \mathbb{Z} -flat of rank $p_{\mathcal{S}}(d)$ provided that $d >$ the maximum of the Castelnuovo-Mumford regularities of the ideal sheaves \mathcal{I}_N and \mathcal{I}_L .*

Proof. Consider the direct image of the exact sequence

$$0 \rightarrow \mathcal{I}_N(d+1) \rightarrow \mathcal{I}_L(d+1) \rightarrow \mathcal{I}_{L \subseteq N}(d+1) \rightarrow 0.$$

Lemma 5.5 and the isomorphism $\mathcal{I}_{L \subseteq N}(d+1) \cong \mathcal{S}(d)$ imply these sheaves are \mathbb{Z} -flat. $d >$ the maximum of the Castelnuovo-Mumford regularities of \mathcal{I}_N and \mathcal{I}_L , so we may apply the Cohomology and Base Change Theorem to conclude $R^i \pi_* \mathcal{I}_{L \subseteq N}(d+1) = 0$ for all $i > 0$, and $\pi_* \mathcal{I}_{L \subseteq N}(d+1)$ is \mathbb{Z} -flat.

$\pi_* \mathcal{I}_N(d+1)$ and $\pi_* \mathcal{I}_L(d+1)$ are also \mathbb{Z} -flat, so

$$\text{rank}(\pi_* \mathcal{S}(d)) = \text{rank}(\pi_* \mathcal{I}_L(d+1)) - \text{rank}(\pi_* \mathcal{I}_N(d+1)).$$

Since we are above the necessary Castelnuovo-Mumford regularities, the right-hand side equals $p_{\mathcal{I}_L}(d+1) - p_{\mathcal{I}_N}(d+1)$. The relevant polynomials satisfy $p_{\mathcal{S}}(d) = p_{\mathcal{I}_{L \subseteq N}}(d+1) = p_{\mathcal{I}_L}(d+1) - p_{\mathcal{I}_N}(d+1)$, so $p_{\mathcal{S}}(d) = \text{rank}(\pi_* \mathcal{S}(d))$. \square

Now that we have formulated the regularity of \mathcal{S} in terms of ideal sheaves on a projective space, we can use Gotzmann regularity to give a bound for Castelnuovo-Mumford regularity of \mathcal{S} .

Lemma 5.7. *If B is \mathbb{Z} -flat, the Castelnuovo-Mumford regularities of \mathcal{I}_N and \mathcal{I}_L are bounded from above by the maximum of the Gotzmann numbers of the Hilbert polynomials of \mathcal{O}_N and \mathcal{O}_L . Furthermore, the Hilbert polynomials of \mathcal{O}_N and \mathcal{O}_L can be expressed in terms of the Hilbert polynomials of \mathcal{O}_B and \mathcal{S} (or equivalently \mathcal{O}_B and \mathcal{G}):*

$$\begin{aligned} \bullet \quad p_{\mathcal{O}_N}(t) &= p_{\mathcal{O}_B}(t) + s p_{\mathcal{O}_B}(t-1), \text{ and} \\ p_{\mathcal{O}_L}(t) &= p_{\mathcal{O}_N}(t) - p_{\mathcal{S}}(t-1) \\ \bullet \quad &= p_{\mathcal{O}_B}(t) + s p_{\mathcal{O}_B}(t-1) - p_{\mathcal{S}}(t-1) \\ &= p_{\mathcal{O}_B}(t) + p_{\mathcal{G}}(t-1). \end{aligned}$$

Proof. The first statement is part of the Gotzmann regularity theorem. For the rest, we have the identifications

$$\begin{aligned} \bullet \quad \mathcal{O}_N &\cong \mathcal{O}_B \oplus \mathcal{O}_B(-1)^s, \text{ and} \\ \bullet \quad \mathcal{O}_L &\cong \mathcal{O}_N / \mathcal{S}(-1) \end{aligned}$$

and the exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_B^s \rightarrow \mathcal{G} \rightarrow 0.$$

□

Finally, we can use this bound to guarantee the representability of the moduli of quasi-splines.

Theorem 5.8. *Assume Y is \mathbb{T} -flat and is a closed subscheme of a projective space bundle $\mathbf{P}(\mathcal{V})$ over \mathbb{T} . Let d be sufficiently large so that $\pi_* \mathcal{S}(d)$ is flat. For instance, $d \geq$ the maximum of the Gotzmann numbers of the polynomials $p_{\mathcal{O}_Y}(t) + s p_{\mathcal{O}_Y}(t-1)$ and $p_{\mathcal{O}_Y}(t) + p_{\mathcal{G}}(t-1)$. The functor on locally Noetherian \mathbb{T} -schemes*

$$Z \mapsto \{(\sigma, \mathcal{S}) \mid \sigma \in \Gamma(Z, \mathcal{S}(d)) \text{ and } \mathcal{S} \in \text{QS}_{p_{\mathcal{O}_Y}, p_{\mathcal{G}}}(Z)\}$$

is represented by $\text{Spec}(\text{Sym}^\bullet(\pi_ \mathcal{S}(d))^\vee)$ over $\text{QS}_{p_{\mathcal{O}_Y}, p_{\mathcal{G}}}$.*

Proof. $\pi_* \mathcal{S}(d)$ is locally free over $\text{QS}_{p_{\mathcal{O}_Y}, p_{\mathcal{G}}}$. A morphism $Z \rightarrow \text{Spec}(\text{Sym}^\bullet(\pi_* \mathcal{S}(d))^\vee)$ produces a point $\mathcal{S} \in \text{QS}(Z)$ as well as a homomorphism over Z

$$\mathcal{S}(d)^\vee \rightarrow \mathcal{O}_Z.$$

Dually we have $\mathcal{O}_Z^\vee \rightarrow (\mathcal{S}(d)^\vee)^\vee$. \mathcal{O}_Z^\vee is canonically isomorphic to \mathcal{O}_Z and $(\mathcal{S}(d)^\vee)^\vee$ is canonically isomorphic to $\mathcal{S}(d)$. So we obtain from the map $\mathcal{O}_Z \rightarrow \mathcal{S}(d)$ our global section σ . This process is reversible, so we are done. □

A Billera-Rose Homogenization

We recall the homogenization procedure of [BR91] in scheme theoretic language. Natural algebraic objects in homogenization and projective compactification are filtered algebras and modules. Given a quasi-spline sheaf S over $A = \text{Spec } \mathcal{A}$, where \mathcal{A} is a filtered algebra, this procedure produces graded module hS over the homogenization $\widehat{\mathcal{A}}$ of \mathcal{A} , and a sheaf ${}^h\tilde{S}$ over the projective closure $\widehat{A} = \text{Proj}(\widehat{\mathcal{A}})$.

The graded components of hS are isomorphic to the degree bounded pieces $S_{\leq d}$ of S . Provided the homogenization $\widehat{\mathcal{A}}$ of \mathcal{A} is isomorphic to $\Gamma_*(\mathcal{O}_{\widehat{A}})$ these graded components and degree bounded pieces are isomorphic to the global sections $\Gamma(\widehat{A}, {}^h\tilde{S}(d))$.

These constructions are compatible with ideal difference-conditions in the sense that if S is defined by $(I_{jk})_{jk}$, then

- hS is defined by $({}^hI_{jk})_{jk}$,
- ${}^h\tilde{S}$ is defined by $({}^h\tilde{I}_{jk})_{jk}$,
- and $\Gamma_*({}^h\tilde{S})$ is defined by $(\Gamma_*({}^h\tilde{I}_{jk}))_{jk}$.

Definition A.1. A filtered \mathcal{O}_Z -algebra \mathcal{A} is a quasi-coherent sheaf of \mathcal{O}_Z -algebras, equipped with a quasi-coherent \mathcal{O}_Z -submodule $\mathcal{A}_{\leq d}$ for each $d \in \mathbb{N}$ such that

- $\mathcal{O}_Z \rightarrow \mathcal{A}_{\leq 0}$,
- $\mathcal{A}_{\leq d} \subseteq \mathcal{A}_{\leq d+1}$,
- $\bigcup_d \mathcal{A}_{\leq d} = \mathcal{A}$, and
- $\mathcal{A}_{\leq d} \cdot \mathcal{A}_{\leq d'} \subseteq \mathcal{A}_{\leq d+d'}$.

Definition A.2. A filtered module over a filtered \mathcal{O}_Z -algebra \mathcal{A} is a quasi-coherent sheaf of \mathcal{A} -modules M , equipped with a quasi-coherent \mathcal{O}_Z -submodule $M_{\leq d}$ for each $d \in \mathbb{Z}$ such that

- $M_{\leq d} \subseteq M_{\leq d+1}$,
- $\bigcup_d M_{\leq d} = M$, and
- $\mathcal{A}_{\leq d} \cdot M_{\leq d'} \subseteq M_{\leq d+d'}$.

Definition A.3. Given a filtered module over a filtered \mathcal{O}_Z -algebra \mathcal{A} , we define the **homogenization**

$${}^hM = \bigoplus_d M_{\leq d} \cdot z^d$$

where z is “dummy variable.” Morally, we think of an element $m \in M$ as $m = m(\frac{x_1}{z}, \dots, \frac{x_n}{z})$ where $\frac{x_1}{z}, \dots, \frac{x_n}{z}$ are (not-necessarily-algebraically-independent) “coordinates” in \mathcal{A} .

Definition A.4. We denote $\widehat{\mathcal{A}} = {}^h\mathcal{A}$. This is a graded \mathcal{O}_Z -algebra, and within $\widehat{\mathcal{A}}_1$ there is an element $z = 1 \cdot z$. If N is a graded $\widehat{\mathcal{A}}$ -module, we denote by $N|_{z=1}$ the module $N/\langle z-1 \rangle \cdot N$. This module is filtered with

$$(N|_{z=1})_{\leq d} = \text{the image of } N_d \text{ under the quotient map.}$$

Proposition A.5. *Homogenization $M \mapsto {}^hM$ is an exact, fully faithful functor from filtered \mathcal{A} -modules to graded $\widehat{\mathcal{A}}$ -modules. Furthermore the assignment $N \mapsto N|_{z=1}$ is a functor from graded $\widehat{\mathcal{A}}$ -modules to filtered \mathcal{A} -modules which is left adjoint to homogenization. The counit*

$$\epsilon: ({}^hM)|_{z=1} \rightarrow M$$

is a natural isomorphism, and the unit

$$\eta: N \rightarrow {}^h(N|_{z=1})$$

is surjective with kernel equal to the saturation $(0 : z^\infty) \subseteq N$.

Proof. These statements simply require checking definitions. \square

Definition A.6. Let \mathcal{A} be a quasi-coherent sheaf of filtered \mathcal{O}_Z -algebras. Write $A = \text{Spec } \mathcal{A}$, and $\widehat{A} = \text{Proj } \widehat{\mathcal{A}}$ for the **projective closure** of A .

Remark A.7. Our treatment differs only superficially from discussions, such as that in [Gro61], on projective closures in which \mathcal{A} is presented as $\mathcal{A} = \mathcal{T}/\mathcal{I}$ for a graded ring \mathcal{T} and a not-necessarily-graded ideal \mathcal{I} . With such a presentation, one defines $\widehat{\mathcal{A}}$ as ${}^h\mathcal{T}/{}^h\mathcal{I}$, where \mathcal{T} and \mathcal{I} are given the filtration from the grading on \mathcal{T} . This way, one only homogenizes submodules of graded modules. Even though there is no meaningful difference in these formulations, Proposition A.5 becomes awkward to state in terms of submodules of graded modules.

Definition A.8. ([BR91]) Let S be a sheaf of quasi-splines on A . Equip S with the filtration

$$S_{\leq d} = \{(g_1, \dots, g_s) \in S \mid g_i \in \mathcal{A}_{\leq d} \text{ for all } i\}.$$

We call hS the **Billera-Rose homogenization** of S .

Lemma A.9. hS is a quasi-spline sheaf on $\text{Spec}(\widehat{\mathcal{A}})$.

Proof. Omitted. \square

One source of the usefulness of homogenization is that it identifies degree d bounded elements with degree d homogeneous elements.

Lemma A.10. As \mathcal{O}_Z -modules, $M_{\leq d}$ is isomorphic to hM_d .

Proof. Omitted. □

Although, hS defines a quasi-spline sheaf ${}^h\tilde{S}$ on the projective closure \hat{A} , it is not always the case that hS can be recovered from ${}^h\tilde{S}$. However, under mild conditions on \mathcal{A} it can, and in this situation questions about S to be completely translated into questions in projective geometry.

Proposition A.11. *If $\widehat{\mathcal{A}} \rightarrow \Gamma_*(\mathcal{O}_{\hat{A}})$ is an isomorphism, then*

- ${}^hS \rightarrow \Gamma_*({}^h\tilde{S})$ for any quasi-spline sheaf S over A , and
- ${}^hI \rightarrow \Gamma_*({}^h\tilde{I})$ for any quasi-coherent \mathcal{A} -ideal I

are too.

Proof. Γ_* is left exact, so we have inclusions $\Gamma_*({}^h\tilde{S}) \rightarrow \widehat{\mathcal{A}}^{\tilde{S}}$ and $\Gamma_*({}^h\tilde{I}) \rightarrow \widehat{\mathcal{A}}^{\tilde{I}}$. The inclusions ${}^hS \rightarrow \widehat{\mathcal{A}}^{\tilde{S}}$ and ${}^hI \rightarrow \widehat{\mathcal{A}}^{\tilde{I}}$ factor through these maps. $z \in \widehat{\mathcal{A}}_1$, and since z is in degree 1, restricting to $D_+(z) \subseteq \hat{A}$ has the same effect as setting $z = 1$. Setting $z = 1$ carries ${}^h\tilde{S}$ to S and ${}^h\tilde{I}$ to I , so $\Gamma_*({}^h\tilde{S})$ is carried onto S and $\Gamma_*({}^h\tilde{I})$ is carried onto I .

Now observe that z is a non-zero divisor on $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^{\tilde{S}}$, and thus a non-zero divisor on any submodule of these. So Proposition A.5 implies both $\Gamma_*({}^h\tilde{S})$ and hS equal ${}^h\tilde{S}$, and both $\Gamma_*({}^h\tilde{I})$ and hI equal ${}^h\tilde{I}$. □

Remark A.12. It is not always the case that $\widehat{\mathcal{A}} \rightarrow \Gamma_*(\mathcal{O}_{\hat{A}})$ is an isomorphism. For instance, consider

$$\mathcal{A} = \mathbf{C}[x, y] / \langle xy, y^2 \rangle$$

with the degree filtration from $\mathbf{C}[x, y]$. Then $\widehat{\mathcal{A}} = \mathbf{C}[x, y, z] / \langle xy, y^2 \rangle$ and $\Gamma_*(\mathcal{O}_{\hat{A}}) = \mathbf{C}[\sigma][x, y, \sigma, z] / \langle \sigma x, \sigma^2, \sigma z - y \rangle$. A representative of σ in the Čech cohomology with respect to the cover $\{U_x, U_z\}$ is

$$\sigma = (0, \frac{y}{z}) \in \mathbf{C}[\frac{z}{x}, \frac{y}{x}] / \langle \frac{y}{x} \rangle \times \mathbf{C}[\frac{x}{z}, \frac{y}{z}] / \langle \frac{x}{z} \frac{y}{z}, \frac{y^2}{z^2} \rangle.$$

One could also consider the scheme A to be the complement of a hypersurface on a non-projectively-normal variety.

If S is computed from ideal difference-conditions both the homogenization hS of S and the sheaf ${}^h\tilde{S}$ on the projective closure \hat{A} can be computed from the associated ideal difference-conditions.

Proposition A.13. *If S is defined by the ideal difference-conditions $(I_{jk})_{jk}$, then*

- hS is defined by the ideal difference-conditions $({}^hI_{jk})_{jk}$, and

- ${}^h\tilde{\mathcal{S}}$ is defined by the ideal difference-conditions $({}^h\tilde{\mathcal{I}}_{jk})_{jk}$.

Proof. The first statement is an immediate consequence of Proposition A.5. The second follows from the first and the exactness of $(\tilde{\cdot})$: localization is exact, and popping out the 0^{th} graded piece is exact. \square

The preceding results establish what is needed from the Billera-Rose homogenization to use it as a tool for studying quasi-splines over affine schemes using projective geometric techniques. However, in what is in some sense the opposite direction, we include the following observation relating quasi-splines on projective schemes defined by ideal difference-conditions and those on their affine cones.

Proposition A.14. *Let \mathcal{S} be a quasi-spline sheaf over a projective \mathbb{Z} -scheme B defined by the ideal difference-conditions $(\mathcal{I}_{jk})_{jk}$. Then $\Gamma_*(\mathcal{S})$ is a module of quasi-splines over the \mathbb{Z} -affine cone $\text{Spec} \Gamma_*(\mathcal{O}_B)$ defined by the ideal difference conditions $(\Gamma_*(\mathcal{I}_{jk}))_{jk}$.*

Proof. Γ_* is left exact. \square

Acknowledgements

The author would like to thank F. Sottile and D. Cox for their encouragement and comments on an early version of this manuscript. The main ideas of this paper were sorted out while the author was at the IHÉS, and we appreciate the the wonderful working environment and hospitality l’Institut provides.

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